Complex numbers in polar form

Recall that the polar form of complex numbers is $r(\cos\theta + i\sin\theta)$ where $r \in \mathbb{R}_+$ and $\theta \in [0, 2\pi)$.

Exercise 2.8

Compute the following:

a. $(1+i)^{14}$

b. $(1-\cos\alpha + i\sin\alpha)^n \quad \text{for } \alpha \in [0, 2\pi], n \in \mathbb{N}$

c. $z^n + \frac{1}{z^n}$ with $z + \frac{1}{z} = \sqrt{3}$

Solution Exercise 2.8

a. $$\begin{aligned}(1+i)^{14} &= \left((1+i)^2\right)^7 \\ &= (1+2i-1)^7 \\ &= 128(i)^7 \\ &= -128i\end{aligned}$$

b. Firstly write the tangent half angle formula as:

$$\tan\frac{1}{2}\alpha = \frac{\sin\alpha}{1+\cos\alpha} = -\frac{\sin(\alpha+\pi)}{1-\cos(\alpha+\pi)}$$

$$\Updownarrow$$

$$-\tan\left(\frac{1}{2}\alpha - \frac{1}{2}\pi\right) = \frac{\sin\alpha}{1-\cos\alpha}$$

We're going to write $(1-\cos\alpha + i\sin\alpha)^n$ in the form $re^{\varphi i}$. First we find φ:

$$\begin{aligned}
\varphi &= \operatorname{arctan}_2(\sin\alpha, 1-\cos\alpha) \\
&= \arctan\left(\frac{\sin\alpha}{1-\cos\alpha}\right) \\
&= \arctan\left(-\tan\left(\frac{1}{2}\alpha - \frac{1}{2}\pi\right)\right) \\
&= -\arctan\left(\tan\left(\frac{1}{2}\alpha - \frac{1}{2}\pi\right)\right) \\
&= -\frac{1}{2}\alpha + \frac{1}{2}\pi \text{ for } \alpha \in [0, 2\pi]
\end{aligned}$$

Next we find r:

$$\begin{aligned}
r &= \sqrt{(1-\cos\alpha)^2 + \sin^2\alpha} \\
&= \sqrt{1 - 2\cos\alpha + \cos^2\alpha + \sin^2\alpha} \\
&= \sqrt{2 - 2\cos\alpha} \\
&= 2\left|\sin\left(\frac{1}{2}\alpha\right)\right| \quad \text{(This step isn't necessary, it helps computing the final answer)}
\end{aligned}$$

Now we can solve the entire equation:

$$\begin{aligned}
(1-\cos\alpha + i\sin\alpha)^n &= 2^n\left|\sin\frac{1}{2}\alpha\right|^n e^{-\frac{1}{2}\alpha n + \frac{1}{2}\pi n} \\
&= 2^n \sin^n\left(\frac{1}{2}\alpha\right) e^{\frac{1}{2}n(\pi-\alpha)}
\end{aligned}$$

c. $z + \frac{1}{z} = \sqrt{3}$

$z^2 + 1 = \sqrt{3}z$

$z^2 - \sqrt{3} + 1 = 0$

$D = 3 - 4 = -1$

$z = \frac{1}{2}\sqrt{3} \pm \frac{1}{2}i$

$z = e^{\frac{1}{6}\pi i} \vee z = e^{-\frac{1}{6}\pi i}$

$z^n + \frac{1}{z^n} = z^n + z^{-n}$

$e^{\frac{1}{6}\pi in} + e^{-\frac{1}{6}\pi in} \vee e^{-\frac{1}{6}\pi in} + e^{\frac{1}{6}\pi in}$ (this right side can be omitted as $\frac{1}{6} = -\frac{1}{6} \cdot -1$)

$z^n + \frac{1}{z^n} = e^{\frac{1}{6}\pi in} + e^{-\frac{1}{6}\pi in} = 2\cos\left(\frac{1}{6}\pi n\right)$